Research Direction: Symmetry-Decomposed Quadrature Using Parity Components

Motivation

- Numerical quadrature often integrates functions over domains possessing reflection symmetry, such as cubes, rectangles, or spheres.
- Such functions can be decomposed into parity components using independent reflections of each coordinate.
- In three dimensions, this produces eight symmetry-related components (even/even/even through odd/odd/odd).

Observation

- For integration over a reflection-symmetric domain, every component containing an odd parity integrates to zero exactly.
- Consequently, only the fully even component contributes to the final integral.

Possible Advantages

- Eliminate analytically known zero-contribution components before numerical integration.
- Reduce floating-point cancellation arising from large odd components whose exact integral is zero.
- Produce a smoother remaining integrand, which may be more favourable for some quadrature methods.
- Restrict integration to a single symmetry region (for example, one octant) while accounting for symmetry analytically.

Possible Algorithm

- Sample all symmetry-related points simultaneously.
- Apply a small Walsh-Hadamard transform (or equivalent parity decomposition) to obtain the symmetry components.
- Discard the components known to integrate to zero.
- Apply the quadrature rule only to the fully even component.

Possible Extensions

- Adaptive quadrature that estimates the energy present in each parity component.
- Recursive parity decomposition, analogous to the recursive structure of the Fast Walsh-Hadamard Transform.
- Extension to higher-dimensional integrals, where the number of parity components naturally becomes 2^n.

Research Questions

- Does explicit parity decomposition improve numerical accuracy?

- Does it reduce floating-point error through reduced cancellation?
- Can it improve convergence of adaptive quadrature schemes?
- Are there classes of integrands for which the computational overhead is more than offset by improved efficiency or accuracy?

Closing Remark

The idea appears to be a natural combination of symmetry exploitation and parity decomposition. Whether it provides practical improvements over existing quadrature techniques is an open question deserving theoretical analysis and numerical experimentation.